

Nonlocal effects in high energy charged particle beams

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Within the framework of the thermal wave model, an investigation is made of the longitudinal dynamics of high energy charged particle beams. The model includes the self-consistent interaction between the beam and its surroundings in terms of a nonlinear coupling impedance, and when resistive as well as reactive parts are included, the evolution equation becomes a generalised nonlinear Schrödinger equation including a nonlocal nonlinear term. The consequences of the resistive part on the propagation of particle bunches are examined using analytical as well as numerical methods.

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I. INTRODUCTION

The thermal wave model (TWM), [1, 2, 3], describes the dynamics of high energy charged particle beams in accelerators. In the TWM approach, the beam is characterised by a complex-valued wave function, which satisfies a Schrödinger-like evolution equation, where the beam emittance replaces Planck's constant, and the intensity of the wave function corresponds to the beam particle density. The Schrödinger potential, which describes the interaction between the beam and its surroundings, can be expressed in terms of a coupling impedance, and due to collective effects, the coupling is a nonlinear function of the beam density. For purely reactive impedances, the TWM equation reduces to the well-known nonlinear Schrödinger equation. However, by including also the resistive part, the evolution equation becomes a generalised Schrödinger equation containing a new term, which is both nonlinear and nonlocal. The modulational instability properties of this new equation have been analysed previously, [4], and have been shown to agree with results obtained using classical approaches, including kinetic effects like Landau damping, [5].

In the present work we consider the longitudinal dynamics of particle bunches under the influence of the coupling impedance. Since the case with a purely reactive impedance is well known, main emphasis is on the situation where the resistive part is included. The dynamical evolution then proceeds as a competition between linear diffraction, nonlinear self-focusing/defocusing, and nonlocal self-steepening. It is found that the bunch is accelerated/decelerated, and the self-steepening effect makes the pulse shape asymmetric with an extended tail, and

eventually a wave-breaking phenomenon can appear on the steepening edge. An approximate solution for the dynamics of soliton-shaped bunches in the presence of a small resistive impedance is found using variational methods, and a perturbation solution for the initial evolution of a particle bunch is given. This illustrates the interplay between the different effects and is qualitatively in agreement with results obtained by other means, as summarised, e.g., in [6]. No stationary solutions with finite particle number are possible, but semi-infinite shocks and pulses with extended wake fields are found. The analytical predictions are confirmed by numerical simulations of the full generalised nonlinear Schrödinger equation.

II. THE GENERALISED NONLINEAR SCHRÖDINGER EQUATION

Within the TWM, the longitudinal dynamics of particle bunches are analysed in terms of a complex beam wave function $\psi(x, z)$, where z is the distance of propagation and x is the longitudinal extension of the particle beam, measured in the moving frame of reference. The particle density, $n(x, z)$, is related to the wave function according to $n(x, z) = |\psi(x, z)|^2$, see [1]. The evolution of the beam is described by the generalised Schrödinger equation

$$i \frac{\partial \psi}{\partial z} = \alpha \frac{\partial^2 \psi}{\partial x^2} + \kappa |\psi|^2 \psi + \mu \psi \int_{-\infty}^x |\psi(\xi, z)|^2 d\xi, \quad (1)$$

where the longitudinal diffraction parameter, α , can be both positive and negative, depending on the phase slip parameter, [7], and the nonlinear parameters, κ and μ , are proportional to the imaginary (reactive) and real (resistive) parts, respectively, of the coupling impedance.

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In the case $\mu = 0$, Eq. (1) reduces to the fundamental nonlinear Schrödinger equation for which a wealth of information is available. In particular, depending on the sign of the product $\alpha\kappa$, the nonlinearity will either counteract ($\alpha\kappa > 0$) or enhance ($\alpha\kappa < 0$) the diffractive broadening. Furthermore, the velocity of the particle bunch is left unchanged, and no asymmetry is introduced on an initially symmetric bunch. Of special interest is the case $\alpha\kappa > 0$, when shape-preserving soliton solutions are possible as a balance between linear diffraction and nonlinear self-focusing effects.

However, the properties of the full Eq. (1) are not known and will be the subject of the present work. In order to see the physical significance of the new term, it is instructive to qualitatively discuss the nonlinear potential, $V(\psi)$, which in the case $\alpha < 0$ is given by

$$V(\psi) = \kappa|\psi|^2 + \mu \int_{-\infty}^x |\psi|^2 d\xi. \quad (2)$$

Consider first the case $\mu \rightarrow 0$ and $\alpha\kappa > 0$, i.e., assume that also $\kappa < 0$. Then, sech-shaped solutions form a well-shaped potential that allows bound states, solitons, to exist. The fundamental soliton solution corresponding to Eq. (1) is

$$\psi = A_0 \operatorname{sech}(ax) e^{-i\delta z}, \quad a = \sqrt{\frac{\kappa A_0^2}{2\alpha}}, \quad \delta = \frac{\kappa A_0^2}{2}. \quad (3)$$

The nonlocal part of the potential introduced by $\mu \neq 0$ contributes a monotonous term to the potential and creates an asymmetry. A qualitative plot of the total potential corresponding to a field shaped as the fundamental soliton (choosing $A_0 = 1$, $a = 1$, and $\kappa = -1$), is shown in Fig. 1, using different values for μ . It is clear that if μ is small, the evolution of an initially soliton-shaped pulse should involve an acceleration in a direction determined by the sign of μ , but the change of the pulse shape can be expected to be slow due to the similarities with the conditions for soliton propagation. For large μ , however, it is obvious that there can be no pulse-shaped stationary solutions, since the total potential then is monotonous, and thus is unable to provide the compression effects needed for confinement. By noticing that the slope of the potential varies over the pulse, and that the more intense parts are accelerated/decelerated more, strong internal pulse dynamics is anticipated. Our subsequent analysis will confirm this intuitive picture.

III. PERTURBED SOLITON DYNAMICS

From the potential picture it is expected that one of the main effects of the nonlocal term is to induce an acceleration of an initially stationary pulse. A more quantitative analysis of this effect can be carried out by investigating the adiabatic evolution of the soliton solution, Eq. (3), in the presence of a small but finite value of μ . This can

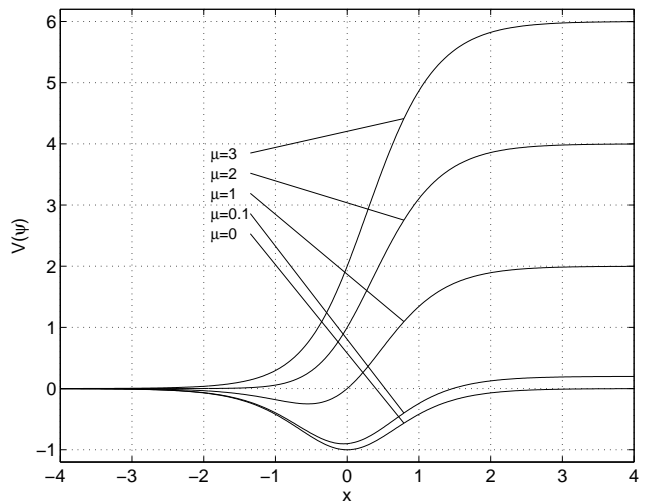


FIG. 1: A qualitative plot of the potential, Eq. (2), for a pulse-shaped field using different values of μ .

conveniently be done using a direct variational approach, see, e.g., [8]. A suitable trial function is

$$\psi_T(x, z) = A \operatorname{sech}[a(x - M)] e^{i[C(x - M) + D]}, \quad (4)$$

where $A(z)$, $a(z)$, $C(z)$, $D(z)$, and $M(z)$ are unknown parameter functions to be determined by the variational procedure. We emphasise that this ansatz function neglects any asymmetric pulse deformations and consequently can model only part of the dynamical evolution. Using Ritz optimisation, these parameter functions can be determined and the following approximate solution is obtained

$$\begin{aligned} \psi_T = A_0 \operatorname{sech} \left(\sqrt{\frac{\kappa A_0^2}{2\alpha}} \xi \right) \\ \times \exp \left\{ i \left[-\frac{2\mu A_0^2}{3} z \xi - \frac{4\alpha\mu^2 A_0^4}{27} z^3 \right. \right. \\ \left. \left. - \left(\mu A_0^2 \sqrt{\frac{2\alpha}{\kappa A_0^2}} + \frac{\kappa A_0^2}{2} \right) z \right] \right\}, \end{aligned} \quad (5)$$

where

$$\xi = x - \frac{2\alpha\mu A_0^2}{3} z^2. \quad (6)$$

This solution is consistent with the classical NLS equation in the sense that the fundamental soliton, Eq. (3), is recovered in the limit when $\mu \rightarrow 0$. The solution in the general case describes a soliton being accelerated in the original frame of reference. The acceleration, γ , is given by $\gamma = 4\alpha\mu A_0^2/3$ and the concomitant shift of the group-velocity is associated with a frequency shift proportional to z .

In order to check the approximate analytical solutions, but also to obtain results in parameter ranges where analytical solutions are not available, Eq. (1) has been solved

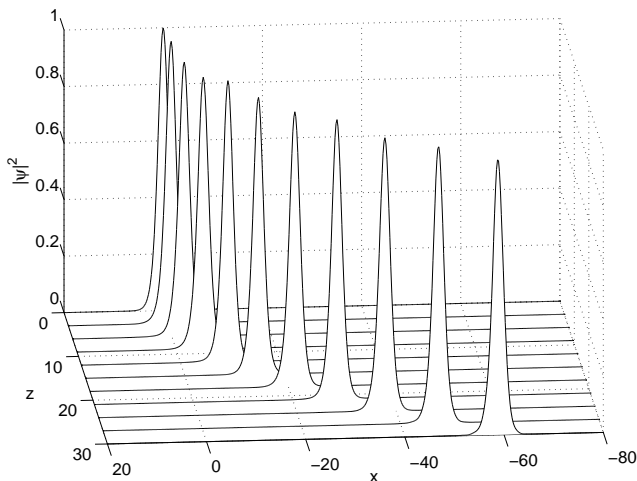


FIG. 2: The numerically obtained dynamics of an initially soliton-shaped pulse using a weak nonlocal term.

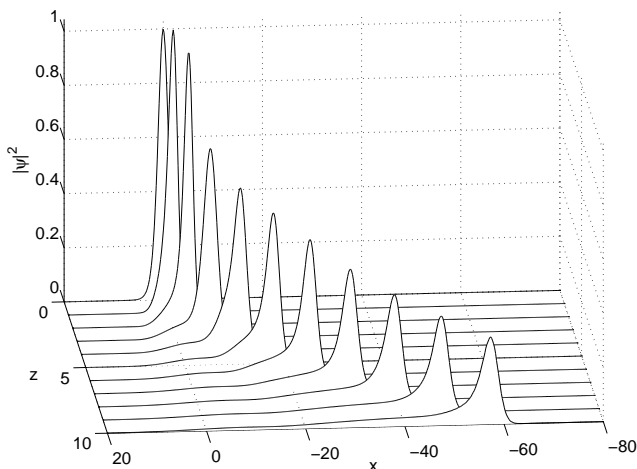


FIG. 3: By increasing the strength of the nonlocal term, strong pulse shape distortion is introduced.

numerically. For this purpose, the standard split-step Fourier method for handling the NLSE is modified to include the effects of the nonlocal term. An example of the dynamics caused by a weak nonlocal term is seen in Fig. 2. The initial pulse, which is centred around $x = 0$, is the fundamental soliton, making a comparison with the variational result simple. The parameters are $\alpha = -1$, $\kappa = -1$, $\mu = 0.1$, and $A_0 = 1$, and the propagation distance is $L = 30$. The pulse is accelerated towards negative x coordinates, but the shape is only weakly distorted. Thus, for small values of μ , the variational result, which predicts the final centre position to be $x = -60$, describes the propagation in an excellent way.

However, for larger values of μ , the asymmetric deformation of the pulse becomes more important and tends to leave an extended tail behind the main pulse. A numerical simulation result of this situation is shown in Fig. 3, where $\mu = 1$. The propagation distance is $L = 10$, mak-

ing the final x -position similar to the example above. It is seen that the internal dynamics of the pulse is significantly stronger; the peak power decreases, and a wake field is developed. Again using the variational result, the final position is expected to be $x = -66.7$, but the decaying peak power makes the potential less steep during propagation, which causes the acceleration to decrease.

Clearly, situations where μ is large can not be analysed by a variational approach based on the soliton ansatz. In order to study the deformation dynamics in some more analytical detail, we will instead use a perturbation analysis.

IV. PERTURBATION ANALYSIS

Although a general solution of Eq. (1) based on analytical methods is not possible, the initial dynamics can be described using a perturbation analysis. For this purpose, Eq. (1) is rewritten as a coupled system in the real amplitude, A , and the phase, θ , of ψ according to

$$\frac{\partial A^2}{\partial z} = 2\alpha \frac{\partial}{\partial x} \left(A^2 \frac{\partial \theta}{\partial x} \right), \quad (7)$$

$$\frac{\partial \theta}{\partial z} = -\alpha \left[\frac{1}{A} \frac{\partial^2 A}{\partial x^2} - \left(\frac{\partial \theta}{\partial x} \right)^2 \right] - \kappa A^2 - \mu \int_{-\infty}^x A^2 d\xi. \quad (8)$$

For an initially unchirped pulse, i.e., $\theta(x, 0) = 0$, the initial amplitude modulation first creates a phase modulation proportional to z , which then generates a subsequent change, proportional to z^2 , of the amplitude modulation. Let us consider the case of the fundamental soliton, Eq. (3), as initial field, since the diffraction and the Kerr nonlinearity then balance each other. Thus, as initial condition we consider $A(x, z = 0) = A_0 \text{sech}(ax)$, where a is related to A_0 according to Eq. (3) and $\theta(x, z = 0) = 0$. Using these in the right hand side of Eq. (8), the initial evolution of θ is obtained, and this solution can then be used to find the lowest order modifications of A from Eq. (7).

However, as found using variational analysis, the pulse evolution is, due to the effects of the nonlocal term, most conveniently described in an accelerated coordinate system. The proper value of the acceleration can be taken from the previous section, but it is also instructive to derive it using an analogy with Ehrenfest's theorem in quantum mechanics. It is straightforward to show that the motion of the mean position, $\langle x \rangle$, of the bunch obeys the equation of motion

$$\gamma_0 \equiv \frac{d^2 \langle x \rangle}{dz^2} = -2\alpha \langle F \rangle, \quad (9)$$

where the averaging is defined according to

$$\langle f \rangle \equiv \frac{\int_{-\infty}^{\infty} f |\psi|^2 dx}{\int_{-\infty}^{\infty} |\psi|^2 dx}, \quad (10)$$

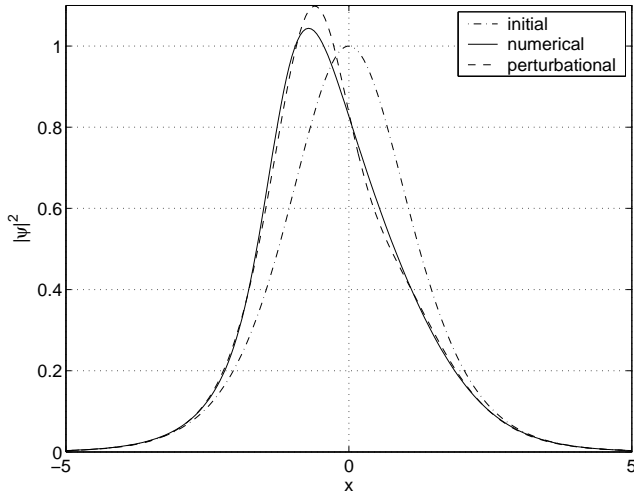


FIG. 4: Comparison of the asymmetric shapes predicted by analysis and numerical simulations, respectively.

and the force is $F = -\partial V/\partial x$. The acceleration obtained in this way is identical to that derived using the variational approach. Thus, $\gamma_0 = \gamma$ and the new coordinate, ξ , is defined according to Eq. (6). The amplitude and phase are then obtained as

$$A = A_0 \operatorname{sech}(a\xi) \times \sqrt{1 + 4\alpha a \mu A_0^2 \tanh(a\xi) \left[\operatorname{sech}^2(a\xi) - \frac{1}{3} \right] z^2}, \quad (11)$$

$$\theta = - \left\{ \frac{\kappa A_0^2}{2} + \frac{\mu A_0^2}{a} [\tanh(a\xi) + 1] \right\} z. \quad (12)$$

The first part of the phase, Eq. (12), does not depend on ξ , and is identical to the phase of the fundamental soliton. The second part is \tanh -shaped, which is due to the form of the nonlocal potential term. As expected from the potential picture, it is found that the amplitude becomes asymmetric.

Due to the limited accuracy, the perturbation analysis can only be applied within a certain propagation distance, obtained from Eq. (11) as

$$4\alpha a \mu A_0^2 z^2 \ll 1 \quad \Rightarrow \quad z \ll 1/\sqrt{|4\alpha a \mu A_0^2|}. \quad (13)$$

Using the same numerical parameters as above, no significant changes in the amplitude are seen within that range. However, as seen in Eq. (13), the application range decreases slowly as μ increases. Thus, a large value, $\mu = 10$, has been used in Fig. 4, where the perturbation analysis is compared with the numerically obtained result after a propagation distance $L = 0.2$. It is seen that the pulse is starting to “lean to the side”, and that the perturbation profile is in good agreement with the numerical result, although its peak value is slightly too large.

V. WAVE-BREAKING

The nonlocal potential term, proportional to μ in Eq. (2), gives rise to a force, $F(\xi) = -\mu|\psi|^2$. This implies that the central parts of the bunch are affected by a stronger force than the wings, and will accelerate/decelerate more. In fact, this is the basic mechanism behind the steepening and the deformation of the bunch. It is clear that after a certain distance of propagation, the high amplitude parts should overtake/be overtaken by the low amplitude parts of the bunch. However, the finite diffraction will prohibit the development of an infinite amplitude gradient, and the “overtaking” between different parts of the bunch instead leads the appearance of oscillations on the amplitude at the base of the steepening side of the bunch. This feature is completely analogous to the wave-breaking phenomenon in nonlinear defocusing Kerr media, [9], with the difference that in the latter case, the corresponding force is an odd function, which implies that the wave remains stationary and that the wave-breaking phenomenon occurs symmetrically on both sides of the pulse.

In order to estimate the length-scale of the wave-breaking phenomenon, the perturbation solution for the amplitude, Eq. (11), can be used. Thus, the order of magnitude of the wave-breaking distance, z_{wb} , is estimated as the shortest propagation length for which the amplitude has a zero. This is easily shown to occur at $z_{wb} = \sqrt{3/(4|\alpha|a\mu A_0^2)}$. It is interesting to note that this approach gives the same result as the one used in [10], which was based on the local velocity shear in the pulse created by the nonlinearly induced chirp, provided the latter is generalised to include the mean acceleration of the pulse.

By increasing the propagation distance in Fig. 4, oscillations on the amplitude will start to occur at the base of the pulse on the steepening side, i.e., close to $x = -3$. Numerically we define the wave-breaking distance as the propagation distance where the amplitude acquires a second maximum. In Fig. 5, the analytical prediction for the wave-breaking distance is compared with the result of the numerical computations. The results show very good agreement, although the numerical results tend to be somewhat larger than predicted. However, since only an order-of-magnitude estimate has been made, the result is quite satisfactory. In particular, the analytic result predicts very well how the wave-breaking distance scales with μ .

VI. STATIONARY SOLUTIONS

As already discussed, the purely reactive case, corresponding to $\mu = 0$, allows a soliton solution, Eq. (3), containing a finite number of particles. In order to investigate whether similar stationary solutions exist also in the general case, we return to Eqs. (7) and (8), which describe the evolution of the (real) amplitude and the

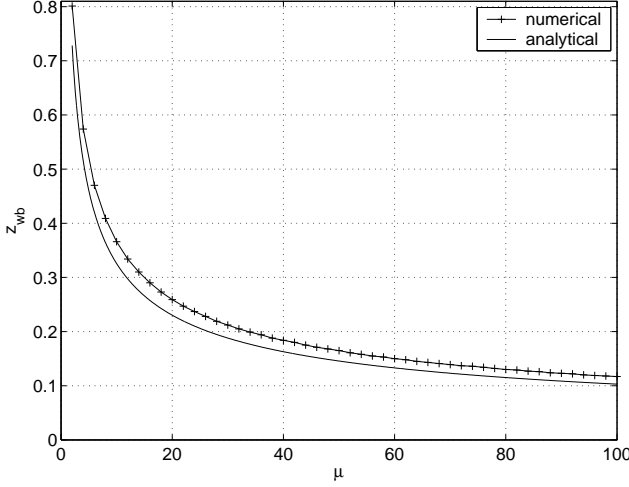


FIG. 5: The wave-breaking distance as predicted by analysis and numerical simulations, respectively.

phase of the wave, respectively. Based on our previous results, we will look for solutions that are stationary in an accelerated frame of reference, i.e., we introduce $\xi = x - \gamma z^2/2$, where the acceleration, γ , now is unknown and has the character of an eigenvalue. Stationarity implies that the amplitude depends only on the coordinate ξ , i.e., $A = A(\xi)$. The phase variation can then be found explicitly, and the system becomes

$$\theta = \theta_0 + C_1 z - \frac{\gamma z \xi}{2\alpha} - \frac{\gamma^2 z^3}{12\alpha}, \quad (14)$$

$$\alpha \frac{d^2 A}{d\xi^2} + C_1 A - \frac{\gamma \xi A}{2\alpha} + \kappa A^3 + \mu A \int_{-\infty}^{\xi} A^2 d\xi' = 0, \quad (15)$$

where C_1 is a constant, which acts as a second eigenvalue. However, by rescaling α , γ , κ , and μ , the equation can be rewritten as

$$\alpha \frac{d^2 A}{d\xi^2} + A - \frac{\gamma \xi A}{2\alpha} + \kappa A^3 + \mu A \int_{-\infty}^{\xi} A^2 d\xi' = 0. \quad (16)$$

In effect, this normalisation sets $C_1 = 1$, and the physical significance of this can be found by letting $\mu \rightarrow 0$ (implying also that $\gamma \rightarrow 0$). The fundamental soliton is then recovered from Eq. (15), and the phase is given by $C_1 z$. Thus, setting $C_1 = 1$ corresponds to normalising with respect to the propagation constant.

Assume that a pulse-shaped stationary solution exists. This implies that $A \rightarrow 0$ as $\xi \rightarrow -\infty$, and asymptotically the field should satisfy the equation

$$\alpha \frac{d^2 A}{d\xi^2} + A - \frac{\gamma \xi A}{2\alpha} = 0, \quad (17)$$

which can be solved in terms of the Airy functions, $\text{Ai}(x)$ and $\text{Bi}(x)$, as

$$A = D_1 \text{Ai} \left(\frac{\gamma \xi - 2\alpha}{\sqrt[3]{2\alpha^2 \gamma^2}} \right) + D_2 \text{Bi} \left(\frac{\gamma \xi - 2\alpha}{\sqrt[3]{2\alpha^2 \gamma^2}} \right). \quad (18)$$

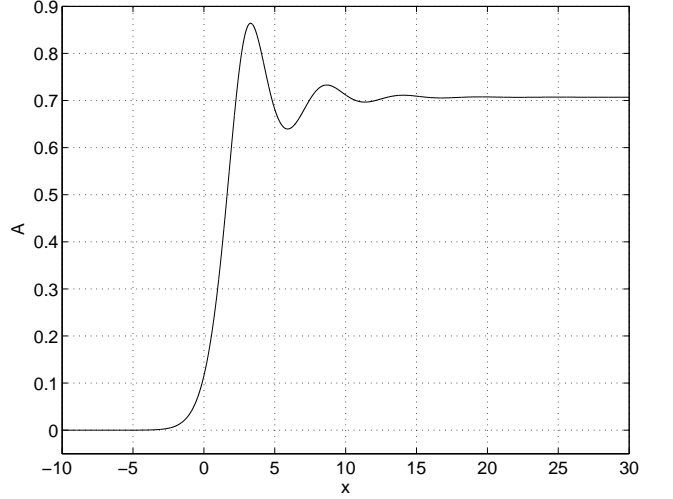


FIG. 6: A numerically obtained step-like solution.

In order to obtain a solution containing a finite number of particles, it is necessary that $\gamma < 0$ and that $D_2 = 0$. For pulse-like solutions, the amplitude of the solution must also vanish as $\xi \rightarrow \infty$ and the corresponding asymptotic equation is

$$\alpha \frac{d^2 A}{d\xi^2} + (1 + \mu W)A - \frac{\gamma \xi A}{2\alpha} = 0. \quad (19)$$

Here, W is the total number of particles, which has been assumed to be finite, and the corresponding solution is

$$A = D_3 \text{Ai} \left(\frac{\gamma \xi - 2(1 + \mu W)\alpha}{\sqrt[3]{2\alpha^2 \gamma^2}} \right) \quad (20)$$

$$+ D_4 \text{Bi} \left(\frac{\gamma \xi - 2(1 + \mu W)\alpha}{\sqrt[3]{2\alpha^2 \gamma^2}} \right). \quad (21)$$

Since the acceleration has already been chosen to be negative, this implies that the asymptotic solution will be the sum of two oscillating Airy functions as $\xi \rightarrow \infty$. However, the total number of particles of such a solution is infinite, and a contradiction has been reached. Thus, we conclude that there are no stationary solutions to Eq. (1) containing a finite number of particles.

On the other hand, if the condition $A \rightarrow 0$ as $\xi \rightarrow \infty$ is relaxed, step-like solutions can be found. By assuming in Eq. (16) that $A \rightarrow A_\infty$ when $\xi \rightarrow \infty$, the integral term is asymptotically equal to $\mu A_\infty^3 \xi$. This term can cancel the term that gives rise to the Airy solutions, provided that $A_\infty = \sqrt{\gamma/(2\alpha\mu)}$. A solution of this type has been calculated numerically by using $\alpha = -1$, $\kappa = -1$, $\gamma = -1$, and $\mu = 1$, and by choosing the amplitude for the asymptotic solution for negative ξ . The result has been plotted in Fig. 6, and it is seen that the predicted value for the asymptotic amplitude, $A_\infty = 1/\sqrt{2}$, is correct.

VII. CONCLUSION

In conclusion, a generalised NLSE, describing the non-linear longitudinal dynamics of high energy charged particle beams in accelerators within the TWM approach, has been analysed using both analytical and numerical methods. It has been discussed in qualitative physical terms how the inclusion of the resistive part of the coupling impedance gives rise to both an acceleration and a deformation of the particle bunch. These effects have been analysed analytically using both a variational analysis and a direct perturbation analysis of the initial dy-

namics, and the results have been shown to be in good agreement with numerical simulations. It has also been shown that for impedances with a large resistive part, the deformation leads to self-steepening and eventually a wave-breaking phenomenon similar to that occurring in nonlinear optics. The scale length for this effect has been estimated and has also been shown to be in good agreement with numerical results. Finally, it has been shown that no stationary pulse-like solutions with a finite number of particles exist for the generalised NLS equation, but semi-infinite shock solutions are possible.

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